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APPROXIMATIONS IN MATHEMATICS REGARDED AS EXACT

BY

G. A. MILLER

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MADRAS

One of the primary facts of the history of mathematics is that approximate results were frequently regarded for long periods of time as accurate. A striking example of this fact is the use of the number 3 as the value of the ratio of the circumference of a circle to its diameter. This value was extensively employed and it is an illustration of a tendency even in mathematics to accept statements as true which could easily have been disproved. In constructing wheels for various purposes in ancient and medieval times it seems reasonable that some of the workmen observed that the given ratio exceeds 3 but their influence on the intellectual world was probably less then than it is to-day, and they were less likely to be familiar with the fact that the number 3 was so widely accepted as the value of this ratio than the workmen of the present day. Hence the persistence of the number 3 for this ratio throws light on the disparity of intellectual influences in ancient times.

The fact that none of the ancient and medieval workmen are now known to have reported that the ratio of the circumference of a circle to its diameter exceeds 3, although closer approximations were also in use, appears less surprising than that the number 3 as the value of this ratio has not been expunged from all of the widely used sacred writings up to the present time. The wise admonition "Prove all things: hold fast to that which is good" was not always completely followed by the writers of our Bible even as regards mathematical questions as may be seen from the following quotation: "And he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits in height thereof; and a line of thirty cubits did compass it round about." If the distance around had been actually measured it would have been found to be almost thirty-one and a half cubits instead of thirty cubits as stated in this quotation.

A large number of other approximate values for the ratio of the circumference of a circle to its diameter were employed by the ancient and the medieval mathematicians. One of these is deduced from the fact that the ancient Egyptians assumed that the area of a circle is equal to that of a square whose side is equal to 8/9th of the diameter of the circle. This is equivalent to assuming that the value of the given ratio is 256/81 which is a much closer approximation than the number 3 and has been used as evidence tending to show that the ancient Egyptians were better mathematicians than the ancient Babylonians since the latter commonly employed the number 3 as the value of this ratio. Many other evidences, however, tend to show that the ancient Babylonians were superior to the ancient Egyptians along mathematical lines. The outstanding mathematical achievement of the latter is that they had a correct rule to find the volume of the frustum of a square pyramid but they were probably unable to prove the accuracy of this rule, judging from their other mathematical attainments.

An approximate value of the ratio of the circumference of a circle to its diameter which is closer than the ancient Egyptian value noted above, and seems to have largely superseded this value in ancient times, was given by Archimedes (287-212) in his "Measurement of the Circle." It was at first explicitly given as an approximation but was later widely assumed to be accurate and is still widely used when only rough approximations are required. Its simple form 22/7 makes its use very convenient and it was rapidly adopted in other countries, being used in China as early as the fifth century of our era. The fact that this ratio is a transcendental number naturally prolonged the period during which various approximations to its true value were assumed to be correct since the ancient and the medieval mathematicians knew nothing about the existence of such numbers and hence were unable to explain the difficulties involved in this particular ratio. The fact that it is transcendental was first established by F. Lindemann in 1882.

The use of the number 3 for the value of the ratio of the circumference of a circle to its diameter appears also in the Koran and it is encouraging that no one is known to have been persecuted on account of his efforts to find closer approximations to the true value of this ratio. The comparatively little interest in mathematical questions on the part of the general public in ancient and medieval times is clearly exhibited by the persistence of the use of the number 3 as the value of this ratio in commonly used sacred

தமிழ்நாடு ஆவணக் காப்பக நூலக

writings as well as by the comparatively small space devoted to this subject in the writings which have come down to us from ancient times. Only about 250 of the several hundred thousands cuneiform texts which are now in the various museums of the world are known to be devoted to mathematical tables or to actual mathematical questions, according to O. Neugebauer's "Vorlesungen uber Geschichte der antiken mathematischen Wissenschaften," volume 1, 1934, page 206.

The great disparity of mathematical knowledge among the people of the same country and living at the same time is a primary fact of the history of mathematics. The study of such works as the "Elements" of Euclid, for instance, is insufficient to disclose the actual state of common mathematical knowledge among the Greeks at the time when Euclid lived. Public education has reduced the disparity of mathematical knowledge in modern times but the enormous number of the courses offered in many educational institutions in recent times tends to increase this disparity again and to lessen the amount of common knowledge even among the educated classes. The transmission of such obvious errors as the one that the value of the ratio of the circumference of a circle to its diameter is 3 in the literature which is highly recommended for its reliability is questionable since it tends to lessen the high regard for truth which should be inculcated whenever possible. Hence Christian nations may well consider the desirability of expunging this error from our Bible.

A primary fact of the history of mathematics is that square units of area and cubical units of volume were adopted in ancient times and this was the first step towards the arithmetization of geometry. Its significance was however not always realized as may be seen from the fact that the author of a geometry which appeared in Germany in about 1400 stated that he had seen many laymen who computed the area of a triangle by multiplying its base by one-half of a side which was not perpendicular to the base.* It has often been asserted that the ancient Egyptians used this inaccurate rule to find the area of a triangle but it is now probable that at least some of them know the correct rule to find this area. When the angle between the two sides is almost a right angle the half product of these two sides gives a fairly close approximation to the correct area. Since the determination of the area of a

* J. Tropfke, *Geschichte der Elementar-Mathematik*, vol. 4, (1922) p. 129.

triangle is fundamental in land measurements it is obvious that this problem attracted early attention.

Many statements which are only approximations but are not expressed as such still appear in the literature of logarithms. For instance, in such a widely used work of reference as the Weber-Wellstein "*Enzyklopädie der Elementarmathematik*" it is stated† that Napier's logarithm of a number is its logarithm to the base $1/e$.

Important facts in this connection are that while logarithms to the base $1/e$ do not differ in the first few decimals from those found in the tables by Napier they are essentially different since the latter were not computed with respect to any base. The statement is also very misleading since the number e was not determined until after the days of Napier. Various writers have assumed that the base of a table of logarithms could be determined by means of the numbers given for the logarithms which appear in this table. This is obviously impossible since an infinite number of bases which differ only slightly would give rise to the same logarithms as those found in such a table to a given number of decimals.

Approximations to the roots of given numbers, especially to square roots, were naturally common among the ancients since the roots of rational numbers are usually irrational and the existence of irrational quantities was first established by the ancient Greeks who, however, did not commonly assume the existence of irrational numbers. A cuneiform Babylonian text contains a rule which is the equivalent of the formula $\sqrt{a^2+b}=a+b/2a$, and rules equivalent to this approximation formula were very extensively used in the ancient mathematical literature, including that of the Greeks. In many cases a and b can be so selected that this formula gives a sufficiently close approximation for certain practical purposes. In the special case of the square root of 2 an unusually close approximation appears in the ancient Hindu literature in the following form:

$$1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$$

This gives the first five decimals correctly and is one of the most remarkable approximations which is known to appear in the ancient Hindu literature, and it has given rise to an extensive literature.

One great difficulty in regard to approximate results in the history of mathematics is that it is frequently impossible to determine

† Vol. I, 4th ed., p. 148 (1922).

definitely whether certain results given by an author are supposed to be accurate or only approximate. A large part of mathematics is devoted to approximations which can often be carried so far that the difference between the true value and the one actually given could not be determined by the available instruments of measure. In elementary mathematics the subject of trigonometry is largely devoted to the study of approximate results. The student is thus led early to the use of approximations and it is often considered unnecessary to state explicitly that the results obtained are only approximately true. The mathematician is thus led to leave to the reader in many cases the decision whether certain results given by him are to be regarded as accurate or only approximate and various critical points are naturally obscured thereby since many readers do not like to assume responsibilities involving thinking.

Even in such simple questions as the proof of the existence of irrational quantities the mathematician insists on a degree of exactitude which can never be reached by measurement while in others, such as the use of tables of logarithms, he is satisfied with approximations which are frequently not very close. This dual character of mathematics naturally complicates its history. The ancient Greeks seems to have been the first to recognize this dual nature of mathematics and to have largely confined their attention to the exact side of mathematics, especially after their discovery of irrational quantities. In applied mathematics, on the contrary, it is commonly necessary to deal with approximations and frequently the data to which mathematical reasonings are applied are in themselves such rough approximations as to make exact methods impracticable. Hence most Greek writers failed to consider such applications.

Plato (429–348) deplored the backward state of solid geometry in his day and said that he was ashamed of it not only on his own account but also on account of all the Greeks. Fortunately, Archimedes (287–212) removed part of this shame of the Greeks by finding the beautiful formulas for the area and the volume of a sphere. However, rough approximation rules continued to be used by the Greeks even after the accurate rules became known. For instance, Heron used a rule to find the volume of the frustum of a cone by assuming that it is equivalent to that of a cylinder of the same altitude whose base is equal to the area of the section at the middle of the cone. Heron used also the correct rule to find this volume and he is supposed to have followed the ancient Egyptians in using the given approximation.

Since our Bible contains very little relating to mathematics it may be interesting to note that what may be regarded as the most astounding error in the history of mathematics, viz., the long continued and widely extended use of 3 as the value of the ratio of the circumference of a circle to its diameter, is found therein and is nowhere corrected therein. The mathematical historian is especially interested in such facts since they help him to form a satisfactory view as regards the disparity of mathematical knowledge, and the difficulty of correcting errors after they have become widely accepted as true. The writer of the present article remembers well when as a child he overheard that the given ratio is 3 and that he faithfully accepted it then as evidence of the superior knowledge of his elders who expressed no doubt as regards the accuracy of the statement, and may never have had any occasion to doubt it thereafter since they then lived in a community where only a few books were read and these were read frequently and with implicit confidence.

ON THE PROPERTIES OF INTEGRAL FUNCTIONS WITH BOUNDED VALUES AT A PRESCRIBED SEQUENCE OF POINTS†

BY

V. GANAPATHI IYER

One of the important problems in the theory of Integral Functions is the determination of integral functions with specified values at a prescribed sequence of points and the investigation of the order and type of such functions. Very few significant results could be obtained without restrictions on the nature of the values assigned as well as on that of the points where the function is to take these values. When the values assigned are bounded and proper restrictions are placed on the prescribed sequence of points it is possible to obtain interesting results in this connection. In the following pages I propose to give a summary of the results and an introductory explanation of the subject-matter of the following seven papers in which I have discussed the above problem in detail :—

(A) On Integral Functions of order one and finite type (*Journal of the Ind. Math. Soc.*, New Series, Vol. 2 (1936), pp. 1-12).

† This formed the Introduction to a Thesis approved for the Degree of Doctor of Science in the Madras University.

(B) On Integral Functions of finite order bounded at a sequence of points. (*Ibid.*, pp. 53-66).

(C) Some theorems on Integral Functions bounded at a sequence of points. (*Proc. Lond. Math. Soc.*, Series 2, 43 (1937), 63-72).

(D) A note on Integral Functions of order 2 bounded at the lattice-points. (*Jour. Lond. Math Soc.*, 11 (1936), 247-249).

(E) On Integral Functions of finite order and minimal type. (*Jour. Ind. Math. Soc.*, New Series, 2 (1936), 131-140).

(F) On the order and type of integral functions bounded at a sequence of points. (*Annals of Math.*, 38 (1937), 311-320).

(G) On effective sets of points in relation to integral functions, (*Trans. Amer. Math. Soc.*, 42 (1937), 358-365).

II. A General Account of the Papers

1. **Definition and Notation.** Let $F(z)$ be an integral function and $M(r, F) = \max. |F(z)|$ for $|z| \leqslant r$. The order ρ, the lower type $l(F)$, the upper type (usually known simply as the type) $k(F)$, are defined by the relations

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r, F)}{\log r};\ l(F) = \underline{\lim_{r\to\infty}} \frac{\log M(r, F)}{r^{\rho}} \leqslant \overline{\lim_{r\to\infty}} \frac{\log M(r, F)}{r^{\rho}} = k(F).$$

When ρ is finite, $F(z)$ is said to be of finite order. For a given ρ, $F(z)$ is to be of minimal, normal or maximal type, according as $k(F) = 0$, $0 < k(F) < \infty$, or $k(F) = \infty$.

1·1 Let $[z_n]$ be a distinct sequence of complex numbers having their sole limit point at infinity. There is no loss of generality in supposing that the numbers $[z_n]$ are arranged according to non-decreasing moduli. We shall speak of the exponent of convergence* of $[z_n]$ as its order, for shortness. Let $\sigma(z)$ denote the canonical product with simple zeroes at $z = z_n$, $n = 1, 2, 3, \cdots$ Then it is known that $\sigma(z)$ is an integral function of order ρ. Let η be the lower bound of numbers h for which

$$\sum_{n=1}^{\infty} \left| 1/\sigma'(z_n) z_n^{\,n+1} \right|$$

converges. We define the index of distribution (I. D.) of $[z_n]$ as max. $(0, \eta)$.

* The exponent of convergence of $[z_n]$ is defined as the lower bound of numbers h for which $\Sigma |z_n|^{-h}$ converges.

1·2. When z_n is real and positive for all n we denote the sequence by $[\lambda_n]$. Let $n(r)$ be the number of λ_n not exceeding r. If $\lim_{r\to\infty} \frac{n(r)}{r^\rho} = D$, we say that $[\lambda_n]$ is a measurable sequence of order ρ and density D. When $\rho=1$, the index of condensation δ of $[\lambda_n]$ is given by †

$$\delta = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \log \left| \frac{1}{\sigma'(\lambda_n)} \right|, \sigma(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

When $\rho\neq 1$, we denote by $\delta(\rho)$ the index of condensation of $[\lambda_n{}^\rho]$ which is of order one and density D.

2. *Papers* (*A*) *and* (*B*): The problems discussed in this thesis are concerned with integral functions which take assigned values $[y_n]$ at a prescribed sequence of points $[z_n]$, Therefore it becomes necessary to consider methods for constructing functions $F(z)$ such that $F(z_n) = y_n$. The usual method is, first, to construct an integral function $g(z)$ with simple zeroes at $z = z_n$, $n=1, 2, \cdots\cdots$ This is always possible by using Weierstrass's products. Then we find a sequence of integers $[p_n]$ such that the series

$$\sum_{n=1}^{\infty} \frac{y_n}{g'(z_n)} \frac{1}{z-z_n}\left(\frac{z}{z_n}\right)^{p_n}$$

converges* absolutely and uniformly for all finite z except at the points $z = z_n$. Then any integral function $F(z)$ such that $F(z_n) = y_n$ is given by

$$\frac{F(z)}{g(z)} = f(z) + \sum_{n=1}^{\infty} \frac{y_n}{g'(z_n)} \frac{1}{z-z_n}\left(\frac{z}{z_n}\right)^{p_n} \quad \cdots \quad (1)$$

where $f(z)$ is an integral function to be determined from the properties of $F(z)$. When $[z_n]$ is of finite order ρ, we can take the canonical product $\sigma(z)$ for $g(z)$ in (1) and get

$$\frac{F(z)}{\sigma(z)} = F(z) + \sum_{n=1}^{\infty} \frac{y_n}{\sigma'(z_n)} \frac{1}{z-z_n}\left(\frac{z}{z_n}\right)^{p_n} \quad \cdots \quad (2)$$

In (A) and (B), I have considered the problem of determining the function $f(z)$ and the sequence $[p_r]$ under suitable hypotheses. I show in theorem 5 of (A) and theorem 7 of (B) that, if $[\lambda_n]$ is a sequence satisfying the conditions

(i) $0<\lambda_1<\lambda_2 \cdots\cdots <\lambda_n\to\infty$;

† For the direct definition of the index of condensation and its properties, see V. Bernstein. Series de Dirichlet, Borel Tracts (1933) 22-27. The index is never negative and is zero if $\underline{\lim}_{n\to\infty} (\lambda_{n-1} - \lambda_n) > 0$.

* It is always possible to find such a sequence of integers. See, Dienes, *Taylor Series* (Oxford), 294—298.

(ii) $[\lambda_n]$ is of order ρ with $\underset{r\to\infty}{\lim} n(r)/r^\rho > 0$;

(iii) $[\lambda_n^{2q}]$ is of finite I. D. where q is an integer given by $\rho \leqslant q < \rho+1$;

and if $[z_n]$ denote the zeroes of

$$\sigma(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^{2q}}{\lambda_n^{2q}}\right) \qquad \dots \quad (3)$$

then, $[p_n]$ in (2) can be replaced by a fixed integer $p>0$ and greater than the I. D. and $f(z)$ by a polynomial of degree $p-1$ at most, provided the integral function $F(z)$ of order ρ is of sufficiently small type and is bounded at the points $[z_n]$. A similar result is discussed in theorem 4 of (B). The conclusions derived from these interpolation formulae in the papers (A) and (B) have been improved upon in the paper (C) but these formulae themselves and the method of establishing them are frequently referred to and utilised in the remaining papers.

2.1. *Paper (C)*: Here three main results are proved. In theorem (1) it is shown that when $[z_n]$ is a sequence of order ρ, $0<\rho<\frac{1}{2}$, and finite I. D. and $\sigma(z)$, the canonical product, is such that $k(\sigma)>0$, the hypotheses (i) $F(z)$ is of order ρ, minimal type; and (ii) $F(z_n) = O(1)$ imply that $F(z)$ is a constant. When $[z_n]$ is replaced by a measurable sequence $[\lambda_n]$ of order ρ and density D for which $\delta(\rho)=0$ the hypotheses (i) $F(z)$ is of order ρ with $k(F) < \pi D \cot \pi\rho$; and (ii) $F(\lambda_n) = O(1)$ imply that $F(z)$ is a constant (theorem 2). Next I prove a result valid for all $\rho>0$. Let $[\lambda_n]$ satisfy the conditions (i), (ii) and (iii) of § 2 above. Let $[z_n]$ be the zeroes of

$$\sigma(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^{2q}}{\lambda_n^{2q}}\right).$$

If $F(z)$ is of order ρ and minimal type and if $F(z_n) = O(1)$ then $F(z)$ is a constant (theorem 3). These three are direct generalisations of the result suggested by Polya that an integral function of order one and minimal type bounded at the points $z = \pm n$, $n = 0, 1, 2, \cdots$, reduces to a constant.

2.2. *Paper (D)*: Here, I prove that an integral function of order two and type less than $\frac{1}{2}\pi$ bounded at the lattice-points reduces to a constant.

2.3. *Paper (E)*: In this paper, I discuss a slightly different problem. Instead of supposing that the I. D. of the sequences considered is finite I connect the values $[y_n]$ with the sequence $[\sigma'(z_n)]$

which measures the I. D. Let $[z_n]$ be a sequence of order ρ, $0<\rho<\frac{1}{2}$ and let $k(\sigma)>0$. Let $F(z)$ be an integral function of order ρ and minimal type and let

$$X_n = \max. \left[|y_n|, \ |y_n/\sigma'(z_n)| \right] = O(1)$$

Then $F(z)$ is a constant.

A similar result holds for all $\rho>0$ when $[z_n]$ denotes the zeros of (3) where the condition (iii) § 2 as to the finiteness of I. D. can be dispensed with. I prove two other results for functions of order one. If $[\lambda_n]$, $0<\lambda_1<\lambda_2 \cdots <\lambda_n \to \infty$, be such that $\lim\limits_{r\to\infty} \frac{n(r)}{r} >0$ and if the function $F(z)$ of order one and minimal type be such that $F(\lambda_n) = O(e^{-\alpha\lambda_n})$ for some $\alpha>0$, then $F(z)\equiv 0$, the index on condensation of $[\lambda_n]$ being zero. If $F(z)$ is of order one and type less than $l(\sigma)$ where

$$\sigma(z) = \Pi\left(1-\frac{z^2}{\lambda_n^2}\right)$$

and if $F(\lambda_n) = O\left(e^{-\alpha\lambda_n}\right)$ however large α may be, then $F(z)\equiv 0$, the index of condensation being merely finite.

2.4. *Papers (F) and (G)*: In paper (F) I discuss results similar to that in (C) but it is assumed that for $\rho=1$, the function in question is bounded at points lying on the four halves of the axes of reference while for $0<\rho<1$ they are assumed to lie on both halves of the real axis. The results obtained are correspondingly more precise. In paper (G), I consider the problem of finding a set $[z_n]$ for which

$$\overline{\lim_{n\to\infty}} \frac{\log|F(z_n)|}{|z_n|^\rho} = k(F) = \overline{\lim_{r\to\infty}} \frac{\log M(r, F)}{r^\rho}$$

where $F(z)$ is any function of order ρ and type *less* than d, ρ, d being two finite positive numbers. As a typical example

$$\overline{\lim_{|m+in|\to\infty}} \frac{\log |F(m+in)|}{m^2+n^2} = \overline{\lim_{r\to\infty}} \frac{\log M(r, F)}{r^2},$$

$$m, n = 0, \pm 1, \pm 2, \cdots\cdots$$

for any function of order two and type less than $\pi/2$.

III. Introduction to the Thesis

1. The essential properties of a single-valued analytic function are determined by the nature and distribution of its singularities. An integral function is characterised by the fact that its sole singularity is at infinity; this must be an essential singularity unless the function reduces to a polynomial in which case the singularity is a

pole of order equal to the degree of the polynomial. Therefore an integral function might be regarded as the simplest generalisation of a polynomial.

1.1. The most characteristic property of a polynomial is that it takes any given finite value at n points, n being the degree of the polynomial. If a transcendental integral function be regarded as a polynomial of infinite degree we should expect that it assumes any finite value at an infinity of points. This is, in fact, true with the modification that we must allow for the possible existence of a single value for which this statement ceases to be true. This is the content of the famous Picard's theorem which states that a transcendental integral function takes any finite value, save possibly one, at an infinity of points.

1.2. A polynomial of degree n takes a given value exactly n times. A similar result can be stated for functions of finite order. Let $[z_n(a)]$ be the sequence of points where the function $F(z)$ of order ρ takes * the value a. Let $E(a)$ be the exponent of convergence of $[z_n(a)]$. Then $E(a)=\rho$ except possibly for a single value a for which $E(a)$ might be less than ρ. This result is due to Borel. The classical theory of integral functions is concerned with the generalisation and detailed examination of results of this type.

1.3. It is always possible to determine a polynomial of degree n which takes $n+1$ given values $[y_s]$, $s=1, 2, \cdots n+1$ at $n+1$ prescribed points $[z_s]$, $s=1, 2, \cdots ; n+1$ and there exists only one such polynomial of degree n. The analogue of this result for integral functions is that given any sequence $[z_n]$ of points and an associated sequence $[y_n]$ of values, it is always possible to find an integral function $F(z)$ such that $F(z_n)=y_n$ (II, § 2). But the solution is no longer unique. It is possible to find an infinity of integral functions satisfying the required conditions. Even if it be required that the solution in question should be of appropriate order and type, it is not possible to state in general whether such solutions exist and even if such do exist whether the solution is unique. But if suitable restrictions are placed on $[z_n]$ and $[y_n]$ it is possible to obtain significant results in this direction. It is this problem that forms the subject-matter of this thesis.

2. The first significant result in this connection was suggested by Polya and proved by Tachakaloff and others (theorem A, paper C).

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* Multiple roots of $F(z)-a$ are to be counted according to their multiplicity.

It states that an integral function of order one and minimal type bounded at the points $z = \pm n$, $n = 0, 1, 2, \cdots$ reduces to a constant. This result was, in fact, the starting-point of the investigations the results of which are embodied in this thesis. A similar result has been proved by Polya for functions of order two bounded at the lattice-points (paper D).

2.1. As a first generalisation of Polya's result, I have considered the question of replacing the points $[\pm n]$ by a general sequence $[\pm \lambda_n]$ $0 < \lambda_1 < \lambda_2 \cdots < \lambda_n \rightarrow \infty$. I prove that when $[\pm \lambda_n]$ is of order one and finite I.D. (11, § 1·1) and $l(\sigma) > 0$, $\sigma(z)$ being the associated canonical product, Polya's result holds when $[\pm n]$ are replaced by $[\pm \lambda_n]$. Next, I have considered the question whether similar results hold for functions of any finite order ρ. The answer is found in theorem 3 (C). It is found necessary to consider points lying on $2q$ lines inclined at equal angles π/q, q being an integer given by the inequality $\rho \leqslant q < \rho + 1$. In fact, if $[\lambda_n]$ is of order ρ and $[z_n]$ are the zeroes of

$$\sigma(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^{2q}}{\lambda_n^{2q}}\right)$$

where $l(\sigma) > 0$ and $[\lambda_n^{2q}]$ is of finite I.D., then a function $F(z)$ of order ρ and minimal type such that $F(z_n) = O(1)$ must reduce to a constant. That, in general, it may not be possible to obtain a significant result of this type with less than $2q$ lines, is seen if we consider the function

$$G(z) = \int_0^a \phi(x) e^{-zx} dx,$$

where $a > 0$ is finite and $\phi(x)$ is L-integrable in $(0, a)$. Here $G(z)$ is an integral function of order one and *therefore of order two and minimal type.* It is bounded on the positive real axis and on the two halves of the imaginary axis; that is, on three lines. Therefore for functions of order two, less than 4 $(= 2q)$ lines will not give a result of the type required.

2.2. It is well known that an integral function of order less than 1/2 exhibits many peculiar properties and behaves more closely like polynomials than other integral functions. I have examined whether these functions exhibit any peculiarities in connection with the properties discussed here. The results are contained in theorems 1 and 2 of (C). I show that if $[z_n]$ is any sequence of order ρ, $0 < \rho < \frac{1}{2}$ and finite I.D. while $k(\sigma) < 0$, then an integral function of order ρ and minimal type reduces to a constant if $F(z_n) = O(1)$. In particular if $[\lambda_n]$ is a measurable sequence of order P and density

$D > 0$ for which $\delta(\rho) = 0$ then the above result holds for any sequence $[z_n]$ for which $|z_n| = \lambda_n$. Again when we actually consider the measurable sequence $[\lambda_n]$ having the above properties, a function $F(z)$ of order ρ and type less than $\pi D \cot \pi\rho$ bounded at the points $z = \lambda_n$ must reduce to a constant, so that, in this case, we get a definite lower bound for the type of functions bounded at the points $[\lambda_n]$. These results along with those of the previous paragraph constitute the main advance on the initial result suggested by Polya.

2.3. It is a characteristic feature of the results mentioned above that the exponent of convergence of the sequence where the function in question is supposed to be bounded, is not by itself sufficient to lead to significant results. We have to suppose in addition that the I.D. is finite, and the density is not too small. The I.D. is a rough measure of the rate of increase of the sequence $[|1/\sigma'(z_n)|]$ which governs the relative distribution of the points of the sequence. The reason why some such additional assumption is necessary will become clear if we remember that, whereas the points at which a function takes a given value form an isolated set, the set of all points where $F(z)$ assumes one or other of the values of the bounded set $[y_n]$ is, in general, not isolated. Let $D_n = [z_{n1}, z_{n2}, \cdots]$ be the set of points where $F(z) = y_n$. What we attempt to do in the problems discussed here is to characterize $F(z)$ by the nature of the sequence $[z_n]$ where z_n is 'picked out' from D_n. It is evident that an arbitrary set $[z_n]$, $|z_n| \to \infty$, picked out from $D_1, D_2, \cdots D_n, \cdots$ will not in general characterise $F(z)$. The point z_n chosen from D_n must be representative not only of D_n but also of the relative situation of D_n with respect to the remaining sets $D_1, D_2, \cdots$. This proper 'picking out' of the representative of each D_n may be regarded as giving rise to the necessity for some hypothesis on the I.D. or the index of condensation.

2.4. Again, besides the above properties of the sequence $[z_n]$, it will be seen from the proof of the results mentioned above that the maximum modulus curves of the canonical product play an important part in the determination of the properties of functions bounded at the points $[z_n]$. When the order is less than half this is not effectively seen since there are circles of arbitrarily large radii over which the minimum modulus is as large as the maximum modulus. But for $\rho \geqslant \frac{1}{2}$ this is no longer true. That is why we have had to assume that the points $[z_n]$ are distributed on one or more lines so that the maximum modulus curve is effectively determined.

2.5. Besides the above results, I prove in paper (D) that a function of order two and type less than $\frac{1}{2}\pi$ bounded at the lattice-points must be a constant. In paper (E) I discuss a slightly different problem. Instead of postulating the finiteness of the I.D. of $[z_n]$ I connect the value $[\eta_n]$ with $[\sigma'(z_n)]$ which measures the I.D. Here also I obtain three results (Theorems 1-3 of paper E). Another result discussed in (E) is closely allied to a theorem due to Faber that for a function of order one and minimal type $[F(n)] \geqslant e^{-\alpha n}$ for any $\alpha > 0$ and for almost all positive integers. I show that if $[\lambda_n]$ is a positive increasing sequence tending to infinity for which $\lim_{r\to\infty} \frac{n(r)}{r} > 0$ and for which the index of condensation is zero, a function $F(z)$ of order one and minimal type cannot satisfy the relation $F(\lambda_n) = O\,(e^{-\alpha\lambda_n})$, $\alpha > 0$ unless $F(z) \equiv 0$. If in addition $[\lambda_n]$ is measurable and has density D while $F(\lambda_n) = O\,(e^{-\alpha\lambda_n})$ however large α be, then $k(F) \geqslant \pi D$ unless $F(z) \equiv 0$.

3. Finally it may be noted that all the results obtained here have, besides their explicit content, a bearing on uniqueness theorems. For instance consider theorem 2, paper (C). Let $[\lambda_n]$ be a measurable sequence satisfying the conditions of the theorem. Then any function of order ρ bounded at $[\lambda_n]$ must have its type not less than $\pi D \cot \pi\rho$ unless it is a constant which is a trivial case. On the other hand, suppose that there are two functions $F(z)$ and $\phi(z)$ of order ρ both of whose types are less than $\pi D \cot \pi\rho$. Then if the difference of their values at $z = \lambda_n$ does not numerically exceed a finite number, the two functions differ only by a constant. That is, there are not essentially two distinct functions with the required properties. All the other results also can be interpreted in a similar manner.

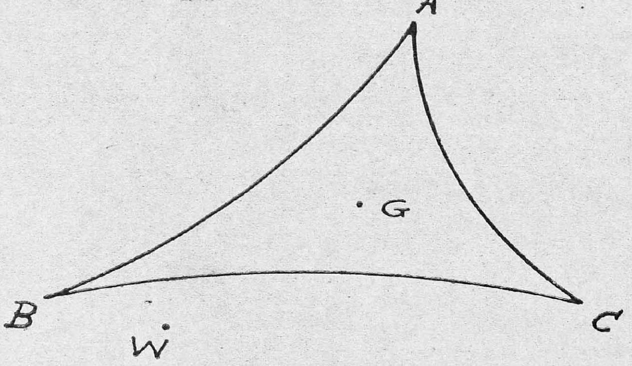
THE META-DELTOID

BY

N. DURAIRAJAN

1. *Introduction*

A curve of order four and class three which has the line at infinity as a bitangent is a biparabola. [R. 1]. It has three cusps and one bitangent. If all the cusps are real, then the points of contact of the bitangent are imaginary. If one cusp is real and two are imaginary, then the points of contact of the bitangent are real. [R. 2.]. The first positive pedal of the biparabola with respect to its unique focus is a conic. If the conic be an ellipse, the biparabola has three real cusps; if it is a hyperbola, one cusp is real and the other two are imaginary; if the hyperbola is rectangular, then the biparabola is rectangular or harmonic [R. 1]. The Deltoid or the three cusped hypocycloid is a limiting case and may be said to correspond to the circle.



THE META-DELTOID
(The position of W is approximate.)

The Deltoid touches the line at infinity at the circular points. If the deltoid is orthogonally projected, we get a biparabola, which

has three real cusps and in which the cuspidal tangents concur at the centroid of the triangle of cusps. I propose to call this curve, as the *meta-deltoid* and to discuss some of its properties. A tracing of the curve is attached.

2. *The equation of the Curve*

If the cuspidal triangle is taken as the triangle of reference, the general equation of a tricuspidal quartic is [R. 2]

$$(lx)^{-1/2} + (my)^{-1/2} + (nz)^{-1/2} = 0$$

when the cuspidal tangents concur at the centroid, the equation becomes

$$x^{-1/2} + y^{-1/2} + z^{-1/2} = 0 \qquad \ldots \quad (1)$$

or $$y^2z^2 + z^2x^2 + x^2y^2 - 2xyz\,(x + y + z) = 0 \qquad \ldots \quad (2)$$

The areal tangential equation is

$$l^{1/3} + m^{1/3} + n^{1/3} = 0 \qquad \ldots \quad (3)$$

or $$(l + m + n)^3 - 27\,lmn = 0 \qquad \ldots \quad (4)$$

or $$p_g^3 = p_1\ p_2\ p_3 \qquad \ldots \quad (5)$$

where p_{g1} p_1 p_2 p_3 are the lengths of the perpendiculars on a tangent from the centroid and the three vertices.

The curve is the isotomic transformation of the Steiner in-conic, whose equation is

$$x^{1/2} + y^{1/2} + z^{1/2} = 0 \qquad \ldots \quad (6)$$

and which cuts the line at infinity at the points of contact of (1) with its bitangent which is $x + y + z = 0$.

3. *The position of the focus*

The foci of (4) are given by substituting $l = z - \alpha$, $m = z - \beta$, $n = z - \gamma$, where z, α, β, γ are the complex numbers representing the focus and the three vertices. [R. 3.]

$$\therefore \quad (3z - \alpha - \beta - \gamma)^3 = 27(z - \alpha)(z - \beta)(z - \gamma) \qquad \ldots \quad (7)$$

This reduces to a linear equation i.e. there is one unique focus, which is given by

$$z = \frac{(\alpha + \beta + \gamma)^3 - 27\alpha\beta\gamma}{9(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \alpha\beta - \gamma\alpha)} \qquad \ldots \quad (8)$$

Remembering that the complex co-ordinate of the centroid and the two Hessian points are

$$g = \tfrac{1}{3}(\alpha + \beta + \gamma)$$

$$h_1 = -\left(\frac{\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta}{\alpha + \omega\beta + \omega^2\gamma}\right)$$

$$h_2 = -\left(\frac{\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta}{\alpha + \omega^2\beta + \omega\gamma}\right) \qquad \ldots (9)$$

where ω is a complex cube root of unity, we get

$$z = \tfrac{1}{3}(g + h_1 + h_2) \qquad \ldots (10)$$

Hence the focus, W, is the centroid of the triangle whose corners are the centroid and the Hessian points of the triangle of cusps. We can write (7) in the form

$$(z - g)^3 = (z - \alpha)(z - \beta)(z - \gamma) \qquad \ldots (11)$$

Equating moduli and arguments, we get the following properties

$$WG^3 = WA \cdot WB \cdot WC \qquad \ldots (12)$$

$$\angle AWG + \angle BWG + \angle CWG = 0 \text{ mod } (2\pi) \qquad \ldots (13)$$

Taking the vertices as origin in succession, we get

$$AG^3 = 3\sigma^2 \cdot AW$$

$$BG^3 = 3\sigma^2 \cdot BW$$

$$CG^3 = 3\sigma^2 \cdot CW \text{ .}$$

$$AG \cdot BG \cdot CG = 3\sigma^2 \cdot GW \qquad \ldots (14)$$

where $\sigma = GS$, S being a focus of the Steiner in-conic. The vector GS is given by

$$GS = g + \tfrac{1}{3}\sqrt{\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta}.$$

The foci of the Steiner in-conic are given by

$$\frac{1}{z - \alpha} + \frac{1}{z - \beta} + \frac{1}{z - \gamma} = 0 \qquad [R. 3].$$

4. *The pedal and orthoptic loci*

The tangential equation of the biparabola referred to the focus as origin is [R. 1] in rectangular cartesians

$$(al + bm)(l^2 + m^2) + n(l^2 + \mu m^2) = 0 \qquad \ldots (15)$$

When $\mu > 0$, the curve is a meta-deltoid. The first positive pedal with respect to the focus is [R. 1.]

$$C_p = x^2 + \mu y^2 - ax - by = 0 \qquad \ldots (16)$$

The orthoptic locus is the conic

$$C_0 = (x-a)[x(1+\mu)-2a]+(y\mu-b)[y(1+\mu)-2b]=0 \qquad \dots (17)$$

The two conics are similar.

5. *Miscellaneous Properties*

(i) If PQR be a maximum inscribed triangle in (16), it is well known that the circles of curvature at P, Q, R intersect the conic again at a point W. and that W, P, Q, R are concyclic. The negative pedal of the ellipse with respect to W is a meta-deltoid whose cuspidal tangents are the lines through P, Q, R, perpendicular to WP, WQ, WR respectively. The point of concurrence of the cuspidal tangents is W′, the point diametrically opposite to W in the circle WPQR.

(ii) If a circle through W cuts the ellipse in X, Y, Z, then the lines LX, LY, LZ perpendicular to WX, WY, WZ respectively are tangents to the meta-deltoid, intersecting at L on the circle WXYZ.

(iii) Consider the circle of curvature at W to the ellipse. The normal to the ellipse at W touches the meta-deltoid at a point, which is the centre of curvature of the ellipse at W.

(iv) The ellipse is tritangent to the meta-deltoid.

(v) The maximum and minimum tangent-chords of the meta-deltoid are parallel to the axes of the steiner circumconic of the triangle ABC and their lengths are two-thirds of the lengths of these axes. This is easily derived from the property that tangent-chords of a deltoid are constant in length and that their orthogonal projections are maximum and minimum for lines parallel and perpendicular to the line of the intersection of the two planes.

(vi) The focus W of the meta-deltoid lies outside the curve, as two of the three tangents from W to the curve are lines joining it to the circular points at infinity.

References

R. 1. *Jour. Ind. Math. Soc.* Vol. 17 (1927) P. 77.

R. 2. BASSET *Cubic and Quartic Curves.* P. 132. 1901 Edition.

R. 3. *Jour. Ind. Math. Soc.* Vol. 17 (1928) P. 223.

NOTES AND DISCUSSIONS

A Note On The Nine Points Circle

THEOREM :—*If $\sum$ be a conic inscribed in a triangle and $\sum'$ the confocal to $\sum$ inpolar to the circumcircle of the triangle, the mutual orthoptic circle of $\sum$ and $\sum'$ cuts orthogonally the nine points circle of the triangle.*

It is well known that in any range (4 line system) of conics

$$\sum_\lambda : \quad \sum + \lambda \sum' = 0$$

there is, in general, a unique member inpolar to a given conic locus. It follows from this that given any pencil (4 point system) of conics

$$S_\mu : \quad S + \mu S' = 0$$

we can associate with every member s_μ of the pencil a unique member $\sum_\lambda$ of the range such that $\sum_\lambda$ is inpolar to S_μ. Thus the parameter λ of the range is in homographic correspondence with the parameter μ of the pencil.

Now, the polar circle S of the triangle, the circumcircle, S′ and the nine points circle S″ belong to a coaxal system. The radical axis of the system (together with the line at infinity) is a circle S‴ whose centre is the point at infinity on the line of centres. Thus, the orthocentre s, the circumcentre s', the nine points centre s'' and the point at infinity on this line, say s''', are in homographic correspondence with the circles S, S′, S″ S‴. But, $(ss', s's''') = -1$. Hence S″ is the harmonic conjugate of S‴ w.r.t. S and S′.

Let $\sum''$ be the harmonic conjugate of the absolute conic $\sum'''$ (consisting of the pair of circular points at infinity of the Euclidean plane) w.r.t. $\sum$ and $\sum'$. Now, the F-conic of any two conics X, Y passes through the points of intersection of pairs of conics like $X + \alpha Y = 0$, $X - \alpha Y = 0$ in the range determined by X and Y. Thus, the F-conic of $\sum''$, $\sum'''$ viz.. the director circle of $\sum''$ passes through the common points of $\sum$, $\sum'$ and is, therefore, the mutual orthoptic circle of $\sum$, $\sum'$.

Now, the conics $\sum$ of the confocal system are in homographic correspondence with the outpolar members of the coāxal system of circles S. Thus, S, S′, S‴ are outpolar to $\sum$, $\sum'$. $\sum'''$ respectively. Hence S″ the harmonic conjugate of S‴ w.r.t. S and S′ is outpolar to the corresponding member $\sum''$ in the confocal system Therefore,

by Gaskin's Theorem the nine points circle S″ cuts orthogonally the director circle of $\sum''$ which is the mutual orthoptic circle of $\sum$, $\sum'$.

In particular, if $\sum'$ consists of a pair of points they are a pair of foci of $\sum$ and are, therefore, isogonal conjugates w.r.t. the triangle. Further, the mutual orthoptic circle of $\sum$, $\sum'$ in this case is evidently the pedal circle of the pair of isogonal conjugates. We have thus the following extension of Feuerbach's Theorem by Shûsakn Ogino. (*vide Tohoku. Math. Jour. Vol.* 43, p. 147.)

If a pair of isogonally conjugate points w.r.t. a triangle are also conjugate w.r.t. the circumcircle, their pedal circle cuts orthogonally the nine points circle of the triangle.

Annamalainagar,
27—11—38.

K. RANGASWAMI.

The Twisted Straight Line

Perhaps the best comment on Narasinga Rao's note (*Mathematics Student*, vol. 6, p. 74) is to point out the simplest application of the idea of the twisted line. The two assertions:

The osculating plane at any point of an asymptotic line on a surface is the tangent plane to the surface;

The osculating plane at any point of a geodesic on a surface contains the normal to the surface;

can both be made without the reservation which careful writers have added that the line in question must not be a straight line. In the case of a straight line the assertions define the plane which is to serve as osculating plane according to the part which the line is considered as playing; there is no contradiction in the fact that the plane which corresponds to the one role is at right angles to the plane which corresponds to the other role.

As an example of the use of the first theorem, if the parameter of distribution of a generator of a ruled surface is p, the angle ν between the normal at the central point of the generator and the normal at the point whose distance from the central point is r is given by $\tan \nu = r/p$. Regarded as an asymptotic line the generator therefore has torsion $p/(r^2+p^2)$, and it follows without further investigation that the curved asymptotic line has torsion $-p/(r^2+p^2)$ and that the Gaussian curvature of the surface is $-p^2/(r^2+p^2)^2$.

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E. H. NEVILLE.

A Note on the Principal Axes of a Conic

1. The necessary and sufficient condition that the straight line whose co-ordinates are (l, m, n) should be a diameter of the conic whose areal-tangential equation is

$$Al^2 + Bm^2 + Cn^2 + 2\,Fmn + 2\,Gnl + 2\,Hlm = 0,$$

is that its pole, i.e., the point

$$(Al + Hm + Gn,\ Hl + Bm + Fn,\ Gl + Fm + Cn),$$

should lie on the line at infinity.

This condition is

$$\sum(A + H + G)l = 0 \qquad \ldots \quad (1)$$

If, further, the line is a principal axis of the conic it is perpendicular to its conjugate diameter for which the condition is:

$$\sum(Al + Hm + Gn)(m - n) \cot \alpha = 0 \qquad \ldots \quad (2)$$

where α, β, γ are the angles of the triangle of reference.

2. If the line (l, m, n) is a principal axis of the conic

$$Al^2 + Bm^2 + Cn^2 = 0,$$

then from equations (1) and (2), above, we see that

$$Al : Bm : Cn = M - N : N - L : L - M, \qquad \ldots \quad (3)$$

where

$$L = (n - l) \cot \beta - (l - m) \cot \gamma,$$
$$M = (l - m) \cot \gamma - (m - n) \cot \alpha,$$
$$N = (m - n) \cot \alpha - (n - l) \cot \beta.$$

For a system of conics touching four straight lines, taking the common diagonal triangle for the triangle of reference, the four straight lines have co-ordinates of the form $(l_0, \pm m_0, \pm n_0)$; and the tangential equation of any conic of the system is

$$Al^2 + Bm^2 + Cn^2 = 0,$$

where

$$Al_0^2 + Bm_0^2 + Cn_0^2 = 0 \qquad \ldots \quad (4)$$

Hence, if the line (l, m, n) be a principal axis of a conic of the system, then from (3) and (4) we see that

$$\sum \{l_0^2 M - N)/l\} = 0$$

i.e.

$$\sum l_0^2 mn(M - N) = 0 \qquad \ldots \quad (5),$$

which is therefore the tangential equation of the envelope of the axes of conics of the system. It is of class three.

In the particular case, when one conic of the system is a circle, its tangential equation is

$$\sum l^2 \tan \alpha = 0.$$

Henec from (4)

$$\sum l_0^2 \tan \alpha = 0 \qquad \ldots \quad (6)$$

Using (6), equation (5) becomes

$$\Big[\sum l \tan \alpha \Big]\Big[\sum l_0^2\, mn \tan \alpha \Big] = 0 \qquad \ldots \quad (7)$$

Hence, the envelope breaks up into

(i) the point ($\tan \alpha$, $\tan \beta$, $\tan \gamma$), which is the centre of the circle and the orthocentre of the triangle of reference ; and

(ii) the parabola

$$\sum l_0^2\, mn \tan \alpha = 0,$$

which touches the sides of the triangle of reference.

For a system of conics through four points, taking the common self-conjugate triangle for the triangle of reference, the four points have co-ordinates of the form (x_0, $\pm\ y_0$, $\pm z_0$) ; and the tangential equation of any conic of the system is

$$\mathrm{A}l^2 + \mathrm{B}m^2 + \mathrm{C}n^2 = 0,$$

where $$x_0^2/\mathrm{A} + y_0^2/\mathrm{B} + z_0^2/\mathrm{C} = 0 \qquad \ldots \quad (8)$$

Hence from (3) and (8), the tangential equation of the envelope of the principal axes of conics of the system is

$$\sum x_0^2\, l/(\mathrm{M} - \mathrm{N}) = 0,$$

i.e. $$\sum x_0^2\, l(\mathrm{L} - \mathrm{M})(\mathrm{L} - \mathrm{N}) = 0, \qquad \ldots \quad (9)$$

which is of class three.

In the particular case, when one conic of the system is a circle, we have, as in (6),

$$\sum x_0^2 \cot \alpha = 0 \qquad \ldots \quad (10)$$

Using (10), equation (9) becomes

$$\Big[\sum l \cot \beta \cot \gamma \Big]\Big[\sum x_0^2\, (l - m)(l - n) \Big] = 0.$$

Hence, the envelope breaks up into

(i) the point ($\tan\alpha$, $\tan\beta$, $\tan\gamma$), the centre of the circle; and

(ii) two points at infinity represented by the equation

$$\sum x_0^2 (l-m)(l-n)=0.$$

From (ii) we see that if one conic of a four point system is a circle, then the principal axes of all conics of the system are parallel—a result which follows from the well-known theorem "*A pair of common chords of a circle and a conic are equally inclined to the principal axes of the conic.*"

PROF. Watson has obviously made a slip on p. 263, Vol. XXII of the *Mathematical Gazette*, where he states that (ii) is a parabola.

3. From the above result we see that if the line $lx+my+nz=0$ is a principal axis of the conic

$$S\equiv(a,\ b,\ c,\ f,\ g,\ h)(x,\ y,\ z)^2=0,$$

then, one conic of the system

$$kS-(lx+my+nz)^2=0$$

must be a circle.

Therefore

$$\frac{kp-(m-n)^2}{a_0^2}=\frac{kq-(n-l)^2}{b_0^2}=\frac{kr-(l-m)^2}{c_0^2},$$

where $p=b+c-2f$, $q=c+a-2g$, $r=a+b-2h$, and a_0, b_0, c_0, are sides of the triangle of reference.

Eliminating k from the above equations, the co-ordinates $(l,\ m,\ n)$ of the line satisfy the equation

$$\begin{vmatrix} (m-n)^2 & p & a_0^2 \\ (n-l)^2 & q & b_0^2 \\ (l-m)^2 & r & c_0^2 \end{vmatrix}=0,$$

which is the tangential equation of the points at infinity on the principal axes.

4. The equation of the principal axes of the conic

$$(a,\ b,\ c,\ f,\ g,\ h)(x,\ y,\ z)^2=0,$$

can be obtained in the form

$$\begin{vmatrix} (Y-Z)^2 & p & a_0^2 \\ (Z-X)^2 & q & b_0^2 \\ (X-Z)^2 & r & c_0^2 \end{vmatrix} = 0,$$

where

$$\begin{aligned} X &= ax + hy + gz, \\ Y &= hx + by + fz, \\ Z &= gx + fy + cz. \end{aligned}$$

For, if x, y, z, be any point on a principal axis, then one conic of the system

$$kS - (xX_1 + yY_1 + zZ_1)^2 = 0,$$

must be a circle.

The equation of the axes can also be written in the form

$$\begin{vmatrix} (X-Y)(X-Z) & q+r-p & \cot\alpha \\ (Y-Z)(Y-X) & r+p-q & \cot\beta \\ (Z-X)(Z-Y) & p+q-r & \cot\gamma \end{vmatrix} = 0.$$

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GLEANINGS

In mathematics we use many easy words: "group," "family," "ring," "simple curve." But these ordinary words of English are given a very peculiar and difficult technical meaning. In fact here is a booby prize definition for mathematics: *Mathematics is the science which uses easy words for hard ideas.*

EDWARD KASNER: *Essay on New Names in Mathematics.*

Two Fundamental Inequalities

I. If x, p, q be all positive, then

$$\frac{x^p-1}{p} \gtrless \frac{x^q-1}{q} \qquad \left(p \gtrless q\right).$$

II. If $x>0$ and $\neq 1$, then

(1) $mx^{m-1}(x-1)>x^m-1>m(x-1)$, $[m<0$, or $m>1]$,

and (2) $mx^{m-1}(x-1)<x^m-1<m(x-1)$, $[0<m<1]$.

The following proofs of these inequalities are based on the calculus and are, on that account, less elementary and less fundamental. It will be seen that the proofs assume the formula $dx^n/dx=nx^{n-1}$ when n is irrational, which is not quite an elementary result. But the proofs now become quite easy.

Proof of I. If $x>1$, then $x^{p-q} \gtrless 1$ for $p \gtrless q$. This inequality is equivalent to $x^{p-1} \gtrless x^{q-1}$ for $p \gtrless q$. Integrating from 1 to x we have

$$\frac{x^p-1}{p} \gtrless \frac{x^q-1}{q} \text{ for } p \gtrless q \text{ and } x>1$$

If $0<x<1$, then $x^{p-q} \lessgtr 1$ for $p \gtrless q$, and $x^{p-1} \lessgtr x^{q-1}$. Integrating from x to 1 since $x<1$, we have

$$\frac{1+x^p}{p} \lessgtr \frac{1-x^q}{q}$$

Hence $$\frac{x^p-1}{p} \gtrless \frac{x^q-1}{q}, \text{ for } p \gtrless q \text{ and } x<1$$

Proof of II. (1) Suppose, in the first place, $x>1$. Then, for all real values of m we have $x^{m-1}>x^{m-2}$. Integrating from 1 to x we have

$$\frac{x^m-1}{m} > \frac{x^{m-1}-1}{m-1}.$$

If $m>1$ or $m<0$ this gives on multiplication by $m(m-1)$ which is positive $\quad (m-1)(x^m-1)>m(x^{m-1}-1)$

that is, $\quad mx^{m-1}(x-1)>x^m-1$;

The first result of II (1) is therefore proved for $x>1$.

To prove the other part of II (1) we note that for $x>1$, $m>1$, $x^{m-1}>1$, which, on integration from 1 to x gives $x^m-1>m(x-1)$; and that if $x>1$, $m<0$, $x^{m-1}<1$, which gives on integration from 1 to x,

$$\frac{x^m - 1}{m} < x - 1,$$

or $$x^m - 1 > m\ (x - 1), \text{ since } m < 0.$$

Hence in either case $x^m - 1 > m\ (x-1)$, and II (1) is proved completely for $x > 1$. The substitution $y = 1/x$, $x > 1$, will show that the same inequality is true for $0 < y < 1$. The inequality II (2) can be proved similarly.

Lahore. } VIDYA PARKASH MALHOTRA.

SOLUTIONS TO QUESTIONS

Question 1699

(A. RANGANATHA RAO):—If O is a given point in the plane of a conic S, prove that :

(i) there exist four points A, B, C, D on S such that the directrix of the unique parabola having third order contact with S at each of these points, passes through O.

(ii) A, B, C, D lie on a circle centre O.

(iii) the circle ABCD is out-polar to S.

(iv) ABCD is the unique cyclic tetrad on S with centre O, apolar to the feet of the normals from O to S.

Solution by K. Rangaswami.

Let σ be a parabola having third order contact with S at a point P and whose directrix passes through O. One pair of opposite vertices of the quadrilateral circumscribed to σ and S is the coincident pair of points PP. Hence the coaxal system determined by the director circle of S and the directrix of σ has P for a limit point.

Now, any circle with centre O cuts orthogonally the directrix of σ; in particular, the unique circle Σ with centre O and out-polar to S also cuts orthogonally, by Gaskin's Theorem, the director circle of S. Thus, Σ belongs to the conjugate coaxal system and, therefore, passes through the limiting point P of the first system. Hence the four points A, B, C, D common to Σ and S are the possible positions for P.

Further if Γ be the Steiner parabola of O w. r. t S, then O lies on the directrix of Γ. Hence the circle Σ is out-polar to Γ. But Σ is also out-polar to S. Therefore, the intersection tetrad of Σ and S is apolar to the contact tetrad of S and Γ i.e., the cyclic tetrad ABCD is apolar to the pedal tetrad of O w. r. t S.

Also solved analytically by the Proposer.

Question 1743

(N. DORAIRAJAN):—If P be any point on the edge CD of a tetrahedron ABCD, show that

(i) the locus of the circumcentre of the triangle PAB is a plane nodal circular cubic;

(ii) the locus of the orthocentre of the triangle PAB is a skew cubic and that it becomes a circle if the lines AB, CD are at right angles.

Solution by C. N. Srinivasiengar.

The first part is identical with Ex. 12. P. 203 in Hilton's *Plane Algebraic Curves.* The locus (i) is evidently a curve lying in the plane bisecting AB at right angles. If O be the mid-point of AB, and if the sphere on AB as diameter cuts CD in H and K, the circumcentre of each of the triangles ABH, ABK is O itself. Hence O is a node on the locus. Next consider any line OL perpendicular to AB. The plane OL, AB meets CD in one point P such that the circumcentre of $\triangle$ ABP lies on OL. Thus OL meets the locus twice at O (since O is a node) and once elsewhere. The locus is thus a plane cubic with a node at O.

To obtain the locus analytically, choose O as origin, and the points A, B as $(0, 0, \pm a)$. The circle with centre at $(x_1, y_1, 0)$ and passing through A and B is given by

$$x^2+y^2+z^2-2xx_1-2yy_1-a^2=0;\ xy_1-x_1\ y=0.$$

By finding the condition that this circle intersects CD given by $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$, we easily get the locus of $(x_1, y_1, 0)$ in the form

$$2\,(l\beta-m\alpha)(ly-mx)(x^2+y^2)=(\beta x-\alpha y)^2+2\,\mathrm{A}\,(\beta x-\alpha y)(ly-mx)+\mathrm{B}\,(ly-mx)^2,$$

where $\mathrm{A}=l\alpha+m\beta+n\gamma$, $\mathrm{B}=\alpha^2+\beta^2+\gamma^2-a^2$.

We thus get a nodal circular cubic.

(2) The orthocentre is the intersection of the plane ABP, the plane through A perpendicular to BP, and the plane through B

perpendicular to AP. As P varies, these three planes evidently form three pencils through AB, the altitude from A on plane BCD, and the altitude from B on plane ACD, such that there is a (1, 1) correspondence between the planes of any two pencils. Hence the locus of the orthocentre is a twisted cubic in general.

Choosing axes etc. as before, the locus is easily found to be

$$x = \{a^2 - (\gamma + nr)^2\}(\alpha + lr) \div d$$
$$y = \{a^2 - (\gamma + nr)^2\}(\beta + mr) \div d$$
$$z = \gamma + nr,$$

where r is the parameter, and $d = (\alpha + lr)^2 + (\beta + mr)^2$. If AB is perpendicular to CD, $n = 0$, and the locus reduces to

$$z = \gamma\,;\ x^2 + y^2 = (a^2 - \gamma^2)^2 w.$$

The locus of the centroid of ABP is evidently a line parallel to CD. Since the nine-points centre is a point of trisection of the join of the centroid and the orthocentre, the locus of the nine-points centre will in general be a twisted cubic.

Question 1746

(MISS S. PANKAJAM):—If x and y be any two given integers and r an unknown integer, find (*a*) the integers r

and (*b*) the number of such values of r

for which

(i) $\binom{x}{r}$ is odd ;

(ii) $\binom{x}{r} - \binom{y}{r}$ is odd ;

(iii) $\binom{x}{r} + \binom{x}{r-2}\binom{y}{2} + \binom{x}{r-4}\binom{y}{4} + \cdots$ is odd

[$\binom{x}{r}$ etc. are the usual binomial coefficients.]

Solution by the Proposer.

(i) We express x in the binary scale so that

$x = 2^m + 2^n + 2^p + \cdots$ to $\sigma(x)$ terms $(m > n > p \cdots\cdots > 0)$.

Now, since $(1+z)^2 \equiv (1+z^2) \pmod 2$

therefore, $(1+z)^{2^2} \equiv (1+z^{2^2}) \pmod 2$

And generally, $(1+z)^{2^t} \equiv 1 + z^{2^t} \pmod 2$ ... (I)

Hence,

$$(1+z)^x \equiv (1+z)^{2^m}(1+z)^{2^n} \cdots (1+z)^{2^p} \cdots \text{ to } \sigma(x) \text{ terms}$$

$$= (1+z^{2^m})(1+z^{2^n})(1+z^{2^p}) \qquad (\text{mod } 2)$$

$$= 1 + z^{2^m} + z^{2^n} + \cdots + \Sigma z^{2^m+2^n} + \Sigma z^{2^m+2^n+2^p} \qquad \ldots\text{(II)}$$

(a) It is obvious that the values of x for which $\binom{x}{r}$ is odd, are the powers of z which occur in II. These powers are numbers of the form $2^m + 2^n + \cdots$ i.e. the numbers which are obtained by replacing any number of units by zeros in the expression of x in the binary scale.

(b) The total number of numbers r (including zero) for which $\binom{x}{r}$ is odd, is evidently $2^{\sigma(x)}$, where $\sigma(x)$ is the number of units in the expression of z in the binary scale.

(ii) Let us express x, y, and r in the binary scale and let λ_{00}, λ_{11}, λ_{10}, λ_{01} denote the sets of places in which there are zeros of both x and y, units of both x and y, units of x against zeros of y, zeros of x against units of y respectively.

Then, from the form of the expressions in the binary scale and from (i), the following properties are evident.

(1) For $\binom{x}{r}$, $\binom{y}{r}$ to be both odd, r has zeros at λ_{00}, λ_{10}, λ_{01}.

(2) For $\binom{x}{r}$ odd and $\binom{y}{r}$ even, r has zeros at λ_{00}, λ_{01}, and has at least one unit in the set of places λ_{10}; and for $\binom{x}{r}$ even and $\binom{y}{r}$ odd, r has zeros at λ_{10}, λ_{00}, and one unit at least in the set of places λ_{01}.

(3) For $\binom{x}{r}$, $\binom{y}{r}$ to be both even, r must have units either at the places λ_{00} or at λ_{01} and λ_{10}.

We could now easily find (b). For, the number of values of r for which $\binom{x}{r} - \binom{y}{r}$ is odd = the number for which either $\binom{x}{r}$ or $\binom{y}{r}$ is odd, and the other even. Direct application of (2) gives the actual result. [(1) and (3) give the number of values of r for which $\binom{x}{r} - \binom{y}{r}$ is even] (a) cannot be solved in the general case. But

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if x and y be given numerical values, x_1, y_1 we can find the actual values of r for which $\binom{x_1}{r} - \binom{y_1}{r}$ is odd.

(iii) I have been unable to solve this for any general x, and y.

Question 1750

(C. N. SRINIVASIENGAR):—Prove the following properties of scrolls:—

(1) The tangent lines of a scroll at points along a generator and at right angles to it generate a rectangular paraboloid, which is closely associated with the locus of normals at points along the generator;

(2) the line of striction of either system of generators of these paraboloids is a straight line;

(3) The osculating quadric along a generator of a scroll always touches the scroll "internally";

(4) The osculating quadric along any generator of a right helicoid is a rectangular paraboloid;

(5) If at points along a generator of a general scroll tangent lines are drawn to the orthogonal trajectories of the curved asymptotic lines, these lines generate a quartic scroll having the given generator as a triple line. The quartic scroll reduces to a cubic in special cases e.g. for a right conoid (excluding the right helicoid).

Solution by V. Rangachariar

Take the central point of the generator as the origin of rectangular co-ordinates, the generator as the x-axis and the normal at the central point as the z-axis. The tangent plane at $(u, 0, 0)$ is $\beta z = uy$ where β is the parameter of distribution. The equations of the tangent line at right angles to the generator are

$$\frac{x-u}{0} = \frac{y}{\beta} = \frac{z}{u}.$$

This line generates the rectangular paraboloid $xy = \beta z$, which is evidently closely associated with the locus of normals along the generator, *viz.* $xz = -\beta y$.

The line of striction of the generators

$$\frac{x-u}{0} = \frac{y}{\beta} = \frac{z}{u}$$

of the paraboloid $xy=\beta z$ is clearly the x-axis. The other system of generators is given by

$$\frac{x}{\beta}=\frac{y-v}{0}=\frac{z}{v}$$

whose line of striction is the y-axis i.e. the tangent line through the central point at right angles to the generator. Similarly for the other paraboloid the two lines of striction are the x-axis and the z-axis.

All the generators of the right helicoid are parallel to the same plane, and hence the osculating quadric which is the limit of the quadric through three consecutive generators is a paraboloid. The right helicoid being a minimal surface, the given generator is at right angles to all the other inflexional tangents to the surface at points along the generator. Hence the osculating quadric is a rectangular paraboloid. It may be noted that since the right helicoid is the only real ruled minimal surface, it is the only ruled surface for which the osculating quadric along an arbitrary generator is a rectangular paraboloid.

Taking the line of striction of the generators of a scroll as the directrix, the equation of the scroll is $\mathbf{R}=\mathbf{r}+u\mathbf{d}$, with the usual vector notation. Using Weatherburn *Differential Geometry* Vol. I, Art. 73 and Art. 24, Ex. 1, the curved asymptotic line cuts the generator at an angle ψ given by the equation

$$\tan\psi=\frac{\mathrm{H}}{\mathrm{A}u^2+\mathrm{B}u+\mathrm{C}-\cos\theta}$$

where

$2\mathrm{AD}=[\mathbf{d},\ \mathbf{d}',\ \mathbf{d}'']$, $2\mathrm{BD}=\kappa\,[\mathbf{d},\ \mathbf{d}',\ \bar{\mathbf{n}}]+[\mathbf{d},\ \mathbf{t},\ \mathbf{d}'']$, $2\mathrm{CD}=\kappa\,[\mathbf{d},\ \mathbf{t},\ \bar{\mathbf{n}}]$ κ and $\bar{\mathbf{n}}$ being the curvature of the line of striction and its unit principal normal, the rest of the notation being as in Weatherburn. We know further that $\beta=\dfrac{\mathrm{D}}{a^2}=\pm\dfrac{\sin\theta}{a}$, and hence

$$\mathrm{H}^2=a^2u^2+\sin^2\theta=a^2(u^2+\beta^2).$$

Taking now the co-ordinate axes as before, the equations of the tangent line perpendicular to the curved asymptotic line are

$$\frac{x-u}{\tan\psi\sqrt{(u^2+\beta^2)}}=\frac{y}{\beta}=\frac{z}{u}.$$

This generates the quartic scroll

$$(xy-\beta z)[\mathrm{A}\beta^2z^2+\mathrm{B}\beta yz+(\mathrm{C}-\cos\theta)y^2]=a\beta y^2(y^2+z^2).$$

For a right conoid, the double line is the line of striction and is at

right angles to all the generators. Hence, $\theta=\pi/2$, $\kappa=0$, and therefore $C=0$, $A=0$. The quartic now degenerates into the cubic scroll

$$Bz(xy-\beta z)=ay(y^2+z^2)$$

and the plane $y=0$.

Solution by the Proposer

Consider a given generator $v=$ constant on the ruled surface

$$x=p(v)+ul(v),\ \ y=q(v)+um(v),\ \ z=r(v)+un(v).$$

At any point along this generator the equations of the tangent line to the surface, in the direction determined by $du/dv=\alpha+\beta u+\gamma u^2$, where α, β, γ are functions of v i.e. constants for the generator, are

$$\frac{x-p-ul}{p'+ul'+l(\alpha+\beta u+\gamma u^2)}=\frac{y-q-um}{q'+um'+m(\alpha+\beta u+\gamma u^2)}=\frac{z-r-un}{r'+un'+n(\alpha+\beta u+\gamma u^2)}.$$

We can eliminate u between these equations by putting each ratio equal to $1/\rho$, and solving for ρ, u, and $\alpha+\beta u+\gamma u^2+\rho u$. We will obtain a quadric surface whose equation is of the form

$$R(K+\gamma R)+Q(\beta R+\alpha Q-P)=0 \qquad \dots \quad (a)$$

where P, Q, R are linear functions of $x-p$, $y-q$, $z-r$ and where k is a constant which does not vanish for a non-torsal generator.

Equation (*a*) *determines an* ∞^3 *of quadrics* (*as* α, β, γ *vary*) *all of which touch the given scroll throughout the length of the given generator.* A particular case is when α, β, γ correspond to the second inflexional tangent at the point (u, v). The quadric obtained in this case is the osculating quadric whose distinguishing property is that the contact with the scroll along the generator is everywhere of the second order.

Consider now the tangential direction at right angles to the generator $v=$ constant. For this direction $du/dv=-F/E$, so that $\beta=0$, $\gamma=0$. The quadric (a) now becomes a paraboloid. If the generators of this paraboloid at right angles to the given line be rotated about it through a right angle, they become the normals to the scroll along the given generator.

One of the lines of striction of the paraboloid is evidently the given generator. I have elsewhere* given the result : if the line of striction of one system of generators of a quadric be a straight line, the same is true about the line of striction of the other system ; further the surface is a rectangular paraboloid i.e. its section at infinity consists of two lines conjugate to the circle at infinity.

* Bull. Calcutta Math. Soc. 28 (1936), 51.

Consider next the orthogonal trajectories of the curved asymptotic lines. If α, β, γ take values corresponding to the curved asymptotic line at (u, v), then for the orthogonal trajectory we have

$$\frac{du}{dv} = -\frac{F(\alpha + \beta u + \gamma u^2) + G}{E(\alpha + \beta u + \gamma u^2) + F} = \frac{a_1 + b_1 u + c_1 u^2}{a_2 + b_2 u + c_2 u^2}$$

where a_1, a_2 etc. are functions of v. Eliminating u from the equations of the corresponding tangent by adopting a method similar to that used for getting equation (a), we get

$$Q^2(a_1 Q^2 + b_1 QR + c_1 R^2) = (PQ - KR)(a_2 Q^2 + b_2 QR + c_2 R^2).$$

This leads to the property (5) in the problem. The quartic scroll reduces to a cubic when $\gamma = 0$ i.e. when $c_1 = c_2 = 0$, in particular for the general right conoid.

To prove (3), choose $x = y = 0$ as the generator, so that the equation of the scroll is

$$2(a_0 x + a_1 y) + (ax^2 + by^2 + 2fyz + 2gzx + 2hxy) + \text{higher terms} = 0.$$

At any point $(0, 0, \lambda)$ the inflexional tangents are given by

$x(a_0 + g\lambda) + y(a_1 + f\lambda) = 0$; $\quad ax^2 + by^2 + 2xy + 2fy(z - \lambda) + 2gx(z - h\lambda) = 0$. The osculating quadric is therefore

$$2(a_0 x + a_1 y) + (ax^2 + by^2 + 2fyz + 2gzx + 2hxy) = 0.$$

For given plane parallel to the tangent plane and close to it, the indicatrices of the scroll and the quadric are identical. Hence the surfaces touch internally* (4) follows from the same property as was used to prove (2). The right helicoid being a minimal surface, a given generator is a line of striction of the osculating quadric along it. Hence the quadric is a rectangular paraboloid.

* For the meaning of, and conditions for *internal* and *external* contact between two surfaces, see my article in *Math. Student* 4, 68–71, also A. Narasinga Rao, *Ibid*, 71–77.

REVIEWS

R. LAMBERT: *Structure Générale des Nomogrammes et des Systemes Nomographiques* 1937 pp. 64, 15 francs (Act. Sc. et Indust. No. 493, Hermann et Cie.)

P. LUCKEY: "*Nomographie*": *Practical Guide for the Design of Graphical Calculators*, 3rd edn., 1937. pp. 108. (Math.-Phys. Bibl. I, Bd. 59/60. Paper covers, R. M. 2. 40)

The titles of these books give an indication of their complementary nature. In the first, though it is well illustrated, the only examples of a practical nature are two which concern finance, and are taken from d'Ocagne's works. In the second, besides sketches for the theory, about 25 problems are solved by fully worked nomograms, several of them in more than one way, and these, save one for compound interest, are concerned with physics and engineering.

In a preface to the first d'Ocagne points out that it is a study in the morphology of the nomogram, and says that the most original part of the work is the investigation of nomographic systems. "Resolving these necessitates the use of adjustments. By a discriminating and systematic treatment the author succeeds in simplifying and facilitating these adjustments .... This work should be followed by a number of others of the same type, by different writers, to expound the various processes, both mechanical and graphical, by which numerical calculations can be simplified and expedited." The fundamental ideas used are contact and degrees of freedom, and for a final classification and enumeration of the systems combinatorial methods are employed. This shows the possibility of relating the teaching of nomograms to the study of a type of Algebra more enlightening than that to which students are usually introduced. The viewpoint here involved may be illustrated by saying that alignment is the simplest case of a superposable plane, and that a dummy axis involves two superposable planes.

It may be of interest to analyse the references given at the beginning of "*Nomographie*". These are all in German, save for three of d'Ocagne's works which are excluded here, as well as six for which details are not given. The frequency distribution then according to size is

Number of pages ...		50		100		150		200	
Frequency of books ...	2		4		7		3		3.

This may give some idea of the part nomograms have played in German education and practice since 1918, when the earliest of these 19 volumes was published. Further references are given to German journals and to one Italian journal, and also to special German publications for nomograms used in chemistry, steel and machine construction, electrical work radiation technique, and aviation, artillery and defence problems.

The opening paragraph of the book, on functional scales from "stream and stress measurements", may be somewhat sophisticated for ordinary students and profoundly suspect with the scholar, but the theoretical aspect

of the subject is not at all neglected. Complaint might be made that the treatment of parallel scale nomograms gets into difficulty over the distances between the scales (so that each of the four practical nomograms of this type given has the middle scale exactly midway between the others!), and there may be a touch of opportunism about the suggestion on page 35 that as functional scales can be taken on the axes of Cartesian diagrams so they may be taken where convenient on any geometrical figure; but on the whole theoretical considerations are set out clearly enough. The author waxes almost lyrical over the qualities of alignment nomograms, though he does preserve his balance. One wonders that he does not feel the heavy overloading of the scales in his nomograms with numerals (on p. 67 elaborate numbering of a scale is repeated on the other side of the line); not trusting to the size of graduations for identification tends to obscure the character of the scale and the general numerical relations shown by the diagram as a whole. The treatment includes the general slide rule, and in the final chapter a functional scale is made to slide in any way over a diagram, this being a particular case of the use of sliding sheets, with an elaborate example of which involving seven variables the book concludes. The only omission noticed is the use of triangular rulings, which arise out of the theoretical treatment in Lambert's book.

Both these books would be of considerable help, each in its own way, in a real reconstruction of mathematical courses such as is needed.

JOHN MACLEAN.

G. H. HARDY AND E. M. WRIGHT: *An Introduction to the Theory of Numbers.* Oxford University Press. (1938) 400 pages. 25 sh.

One can count on one's fingers the number of books on the Theory of Numbers in the English language, and hence any new book on the subject is very welcome. When it comes from the hands of such an authority on the subject and inspiring writer as Professor G. H. Hardy, with the collaboration of Professor E. M. Wright, an expert worker in the field, every student of Number Theory will be simply delighted. Besides, this work has many features which secure for it a unique and prominent place among books on the subject.

The book is the outcome of lectures delivered by the authors in various Universities and offers an introduction to almost all the branches of Higher Arithmetic. By easy stages, the authors take the reader through various regions here and there giving glimpses of the many majestic peaks of the Number Theory, conquered or yet to be conquered. Whenever there is an opportunity, readers are taken on excursions to the allied fields of interest. The Chinese game of Nim, Bachet's problem of the weights, the construction of the 17-sided polygon, the Konig-Szuc's problem on the path of rays reflected on the sides of a square, etc. are discussed in detail. There is a digression even on a historical puzzle in one place. The book abounds in unsolved problems of intense interest. When the authors give here and there big numbers, saying "This is the greatest known prime', 'this is the greatest number proved to be composite, but whose factors are unknown', 'this is the value of p (14031)' and so on, it produces a thrill in us.

Another interesting feature of the book is that several different proofs of varying interest are given for many important results, one result receiving as many as four different proofs. When an alternative proof requires an acquaintance with a different field, the authors take the trouble to develop the relevant portions of the alien field also.

Each chapter contains notes at the end, giving references to original sources, further developments in the topic, and sometimes a summary survey of the frontiers of the field. At the end a list of books for further reading and an index of special symbols and terms are also given.

The following rough summary of the contents of the book will give an idea of the variety and richness of the 'programme'.

Three Chapters (I, II, XXII) are devoted to the study of prime numbers. The main topics considered are the fundamental theorem of Arithmetic, Tchebychef's results on the order of the number of primes, Merten's theorem on $\Sigma\, 1/p$ and the numbers of Fermat and Mersenne.

In Chapter III, the properties of Farey series and Minkowski's theorem on the number of lattice points in a closed curve are established. Chapter XXIV contains proofs for three theorems of Minkowski concerning sets of n linear forms in n integral variables with real coefficients.

The authors devote chapters IV, X, XI to the study of irrational numbers and continued fractions. In Chapter X, elementary properties of simple continued fractions (periodic and nonperiodic) are investigated. Further, it deals with 'equivalent numbers', the series of Fibonacci and Lucas, and approximation by convergents. In Chapter XI, orders of approximation, the existence of transcendental numbers, the transcendence of e and π, and some theorems on continued fractions with bounded quotients are discussed.

Chapters V, VI, VII, VIII are concerned with congruences and residues. They contain proofs for theorems of Fermat, Wilson, Lagrange, Von Staudt, and Leudesdorf, the law of quadratic reciprocity, and Bauer's identical congruence; and their consequences.

Besides the usual discussion about the representation of numbers by decimals, Chapter IX contains 'tests for divisibility, certain interesting irrational numbers represented by decimals, some theorems on numbers with missing digits, and results on normal numbers.'

The study of quadratic fields occupies Chapters XII, XIV and XV. The fields $k\,(i)$ and $k\,(\rho)$ are studied in detail, and a few other fields are also considered in short. The fact that unique factorisation theorem is not universally true is clearly brought out. The finitude of 'Euclidean' and 'simple' quadratic fields is discussed and settled when the proof is easy. The proof for Lucas' test for the primality for the Mersenne number, M_{4n+3} is also given.

In Chapter XIII the authors consider the Fermat's equation $x^n+y^n=z^n$, when $n=2$, 3 and 4. Richmond's theorem on the expression of a number as the sum of rational cubes is proved, and the equation '$x^3+y^3=u^3+v^3$' is completely solved.

In Chapters XVI, XVII, XVIII, the well-known arithmetic functions $\phi(n)$, $\mu(n)$, $d(n)$, $\sigma(n)$, $r(n)$ etc. are considered. Their generating functions, and their maximum, minimum and average orders are investigated. Many well-known identities including Ramanujan's are given. The multiplicative properties of the sums of Gauss, Ramanujan, and Kloostermann are established.

The theory of Partitions is considered in Chapter XIX. Several well-known indentities in the theory due to Euler, Jacobi, and Roger and Ramanujan are proved. Certain congruence properties of the partition function due to Ramanujan are also established.

Waring's problem and the related topics form the subject-matter for Chapters XX and XXI. Chapter XX contains three of the four proofs given in the book for the theorem on the expression of a number as the sum of two squares, and three proofs for Lagrange's theorem on four squares. In Chapter XXI, upper bounds for $g(3)$ and $g(4)$, and lower bounds for $g(k)$ and $G(k)$ are evaluated. Further the function $v(k)$ (for the sums affected with signs) and Tarry's problem are alsc discussed.

Chapter XXIII deals with Kronecker's theorem on the simultaneous approximation of given irrationals to arbitrary numbers.

The book does not demand any great mathematical equipment on the part of the reader. From the Indian standpoint one can say that any intelligent mathematics graduate will be able to understand almost the whole of the book without outside help, while about threefourths of the book will be intelligible to a student of a degree class, having aptitude for mathematics.

The book is an intensely fascinating and very inspiring one. There is no doubt that it will inspire numerous youngmen to become ardent devotees to the 'queen' of the 'queen of all the sciences', and stimulate the interest of workers already in the field. Hence it is superfluous to say that every college library should possess at least one copy of this book of extraordinary charm.

The printing and get-up are attractive. I was able to note only one misprint: page 203, '$2^{p-1} \equiv 0$' instead of '$2^{p-1} \equiv 1$'.

I wish to draw the attention of the authors to one or two minor points. Out of the four symbols borrowed from formal logic, apart from the symbol '$\longrightarrow$', the other three are used only in a dozen places or so. In view of the fact that keeping the number of new symbols at its minimum will be conducive to the easy reading of a book, I venture to suggest the omission of the above-mentioned three symbols in the future editions of the book.

The other suggestion is a general one. Is it not against the spirit of Mathematics to continue to call theorems by wrong names even when it has been established that they are wrong? If all the authors of books in Mathematics make up their minds to rectify the mistakes in current names of results, everything may be set alright in a few decades. Why should not authors rename in their books 'Farey series', 'Pell's equations', 'Cardan's solution' etc., at the cost of temporary inconvenience, and act up to the spirit of the 'exact science'?

S. S. PILLAI.

MILNE THOMSON: *Theoretical Hydrodynamics.* Macmillan Co. London 31/6 net.

This book on Hydrodynamics by Prof. Milne-Thomson is a welcome addition to the small number of comprehensive treatises on the subject now available in English, especially in view of its several distinguishing features.

The treatment is based exclusively on vector methods and notation; the advantage of their use in deriving the general equations and in economising space besides giving a better physical insight into the phenomena discussed is specially seen in the chapters dealing with the Equation of Motion and Viscosity. And a very extensive use is made of the complex variable for the discussion of various problems of fluid motion.

One of the earlier chapters is devoted to the properties of vectors which are introduced without any reference to systems of co-ordinates, the subject-matter being arranged in such a way as to help those to whom vector manipulation may be unfamiliar. Again there is a chapter dealing the theory of functions of a complex variable up to contour integration and conformal representation. Thus in respect of the mathematical apparatus required, the book is fairly self-contained.

Two dimensional motion occupies some nine chapters in the book, separate chapters being devoted to general properties, streaming motion, aerofoils, sources and sinks, moving cylinders, Schwarz-Christoffel theorem and applications, jets and currents, and the wake behind a cylinder. One consequence of this arrangement is that allied problems come under different heads and could be discussed as illustrations of different methods. For instance, instead of treating all elliptic cylinder problems with the help of elliptic harmonics chosen to satisfy the necessary boundary conditions, streaming motion past an elliptic cylinder is obtained from the case of a circular cylinder by a Joukowski transformation, and the result translated into elliptic co-ordinates for the purpose of easy description of the flow; while the complex potential for a moving elliptic cylinder is obtained by a simple application of Morris's general method which is applicable whenever the co-ordinates of a point on the cross-section of a cylinder can be expressed in terms of a parameter. Here as elsewhere in the book is incorporated some account of recent work e.g. generalisations of Blasius's formula and Kutta-Joukowski theorem.

The chapter on the Wake includes an account of Levi-Civita's method of determining the flow past an obstacle, making this elegant method available for the first time in an English book. Two dimensional wave motion, which is the subject of chapter XIV, is all based on Cisotti's equation satisfied by the complex potential for waves of small height. There is a departure from the traditional presentation in the treatment of the motion of spheroids. Taking account of axial symmetry, the motion is studied with the help of the complex variable by means of the transformation $x+i\omega = c \sinh \xi$ and $x+i\omega = c \cosh \xi$ for the cases of the planetary and ovary spheroids. The quadratic transformation $x+i\omega = c\xi^2$ similarly gives the motion of a paraboloid of revolution.

In the traditional formulae giving the motion of the Ellipsoid, there is the unnecessary occurrence of the number 2 which is due to the notation

employed. This is easily avoided if the notation is slightly altered by taking α, α_0 etc. to represent half their usual values so that $\alpha_0+\beta_0+\tau_0=1$; and the formulae gain considerably in appearance.

It is stated in the preface that the book has grown out of the author's lectures to classes in the Royal Naval College. The range covered is that usual in the introductory text-books, including chapters on Vertex motion and Viscosity. The exposition is clear and methodical, the needs of the student being borne in mind throughout. The book contains numerous illustrative diagrams and there are over 500 exercises collected into sets of examples at the end of each chapter. For easy reference, a detailed table of contents and a complete index are provided.

The book is thus assured of success as a text-book—though the price 31 s. 6d. is rather forbidding.

T. S. N.

Maurice CURIE et Maurice PROST: *Necessaire Mathematique,* Herman Fils, 1937; pp. 116; Actualites Scientifiques et industrielles No. 502 Price 20 fr.

The University of Paris offers a one year course for the award of a 'Certificat d'Etudes Physiques Chimiques et Biologiques' (briefly P. C. B.), this certificate being demanded of those who seek admission in the Faculty of Medicine. The revised programme of the P. C. B. presupposes a certain level of basic "Necessary Mathematics" which this book seeks to provide.

The book is intended also to provide the "necessary mathematics" for the S. P. C. N.—'Certificat d'Etudes Superieures Physiques Chimiques et Sciences Naturelles.' The level of mathematical equipment for this is somewhat higher and accordingly the additional matter intended for this course is printed in slightly smaller type. For example, the addition formulae trigonometric functions, the derivates of inverse functions, and many of the proofs appear in small type.

In order to make the unpleasant mathematical pill palatable for the P. C. B. there is often an appeal to intuition in preference to a rigorous proof.

The several chapters deal with Algebra up to the solution of the quadratic, Trigonometry, graphs, calculus (Differential and integral including Maclaurin's Theorem and areas and volumes) and differential equations. There are 3 appendices dealing with small errors, probability and with dimensions in physical quantities.

A. NARASINGA RAO

BOOKS RECEIVED FOR REVIEW

Tsurusaburo TAKASU: *Differentialgeometrien in den Kugelraümen*, Bd. I Konforme Differential-Kugelgeometrie von Liouville und Möbius pp. 458. Maruzen Company, Ltd., Tokyo, Price 7·5 R. M.

Raziuddin SIDDIQI: *Lectures on Quantum Mechanics*, Vol. I, 1938 pp. 293, Osmania University, Hyderabad (Deccan), Price Rs. 6.

Raymond Clare ARCHIBALD: *Outline of the History of Mathematics, 4th Ed.* Mathematical Association of America 1939, Price 50 cents.

Wilhelm BLASCHKE: *Ebene Kinematik*, Hamburger Mathematische Einzelschriften 25 Heft, 1938, pp. 56, B. G. Teubner.

ANNOUNCEMENTS AND NEWS

Mr. Sulaiman Mahomed Kerawala, M.A., (Cantab.) Lecturer in Mathematics, Muslim University, Aligarh, has been elected a life member of the Society.

Mr. C. T. Rajagopal, M.A., Lecturer, Madras Christian College, has been elected a member of the Society.

The Hyderabad Conference in December 1939

An invitation to conduct the Eleventh Conference of the Indian Mathematical Society at Hyderabad (Deccan) on the 21st, 22nd and 23rd of December 1939 has been received from the Osmania University and accepted by the Society. The 10th session of the All-India Oriental Conference meets also in Hyderabad at the same time (20th to 23rd December) and the social functions and excursions of the two conferences will be arranged in common.

An influential reception committee has been constituted with the Pro-Vice-Chancellor of the Osmania University as Chairman. The local committee which will be in charge of the arrangements regarding the Conference session consists of Dr. M. R. Siddiqi, M.A., Ph.D. (Chairman), Mr. Khaja Mohinuddin, M.A. and Mr. Venkatachari, M.A.

The following programme has been provisionally fixed:

Thursday 21st December—

Forenoon: Opening ceremony, and the inaugural address.

Afternoon: Symposium on Generalised Geometry (including Relativity and Field theory); Popular lecture.

Friday 22nd December—

Forenoon : Reading of papers.

Afternoon : Symposium on Waring's problem ; Popular lecture.

Saturday 23rd December—

Forenoon : Reading of papers.

Afternoon : Business meeting ; Discussion on the Teaching of Mathematics ; Popular lecture.

Those wishing to contribute papers to the Conference are requested to send them along with 2 brief abstracts of each to A. Narasinga Rao, Annamalai nagar so as to be received by the middle of October.

The 27th session of the Indian Science Congress will be held at Madras from the 2nd to the 8th January 1940. Prof. A. C. Banerji, Head of the Mathematics Department, Allahabad University, Allahabad, has been nominated to be the President of the Mathematics Section. Papers should be sent to the General Secretary, Indian Science Congress Association, 92 Upper Circular Road, Calcutta, so as to reach him by the 15th September 1939 along with 3 typed abstracts of each.

The third session of the Indian Statistical Conference will meet in Mysore on the 4th and 5th January 1940.

Mr. S. Subramaniam, M.A., Lecturer in Mathematics, Annamalai University, has been appointed Statistician to the Economic adviser to the Government of India in place of Dr. P. V. Sukhatme, M.A. Ph.D. (London) who has accepted the post of Assistant Professor of Vital Statistics in the All-India Institute of Public Health and Hygiene, Calcutta.

QUESTIONS FOR SOLUTION

1770. (D. R. KAPREKAR)—The fractions $1/487$ and also $1/(487)^2$ when expressed as decimals have the same number of recurring figures namely 486. Required to find all numbers N such that $1/N$ and $1/N^2$ have the same number of recurring figures in their decimal representation.

1771. (KAIKOBAD F. VAKIL)—Taking the given number 37, 781,456, 166, 601, 770 as step number zero, find after what number of steps we shall get a perfect square by adding to it 50, 952, 413, 380, 206, 183 ; 50, 952, 413, 380, 206, 185 ; 50, 952, 413, 380, 206, 187 ; etc. successively which are in A.P. with a common difference 2.

e.g.	37781456166601770 50952413380206183	Step Number 0.	
	88733869546807953 50952413380206185	...	(1)
	139686282927014138 50952413380206187	...	(2)
	190638696307220325	etc. ...	(3)

1772. (B. R. VENKATARAMAN)—C_i $(i=1, 2, \cdots 4)$ are 4 circles in a plane and S_i $(i=1, \cdots 4)$ are the four circles which are respectively orthogonal to sets of three circles chosen from C_i. If the inverses of a point P w. r. t. the circles C_i are concyclic, prove that the inverses of P w. r. t. the circles S_i are also concyclic.

1773. (G. V. RAMACHANDRAN and B. RAMAMURTI)—l_1 l_2 l_3 l_4 are four straight lines in a plane. L_{rs} is the point of intersection of the lines l_r l_s. If P be any point in the plane show that

(1) there is a conic Σ touching the four lines, for which PL_{21}, PL_{24} and PL_{31}, PL_{34} form two pairs of conjugate lines

(2) the points P and L_{14} are conjugate with respect to Σ

(3) If PQ, PR be any pair of conjugate lines of Σ

$$P(QL_{12}RL_{34}) = P(QL_{13}\ RL_{24})$$

(4) Given the pairs of points A_1A_2, A_3A_4 and another point P all lying in a plane, the pairs of lines PQ, PR such that

$$P(QA_1\ RA_2) = P(QA_3\ RA_4)$$

are conjugate with respect to a fixed conic Σ touching the lines A_1A_3, A_1A_4, A_2A_3, A_2A_4.

1774. (S. S. PILLAI)—n beads are connected in the form of a chain by means of $n-1$ links of which r are detachable. By removing at most t of these detachable links at a time, we should be able to form groups containing any number of beads from 1 to n. Let $f(t, n)$ denote the least value of r so that the above may be possible. It is easy to see that

$$f(1, n) = [(n+1)/2] \text{ and } f(t, n) = t, \text{ when } t = [(\log n)/(\log 2)] + 1.$$

What is the value of $f(t, n)$ for intermediate values of t?

THE

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and others.

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Page	*line*	*for*	*Read*
68	9B†	13345678987654321	12345678987654321
74	6B	$k=0$	$\kappa=0$
77	2	$\sum \frac{z^r}{r!} = 0$	$\sum \frac{z^r}{r!}$
83	12	$\frac{1}{2}$ AB · DE	$\frac{1}{2}$ AB · AE
99	17-32	Remove the first 5 sentences of § 3 (ending with the words "Circular points at infinity") and replace them by the following:	

"Any two concurrent tetrads of the Miquel-Clifford configuration have 2 circles in common. Let P and Q be the limiting points of the coaxal system they determine. The two cyclics associated with these tetrads have in common as we have seen, the 18 points of the configuration and also P, Q. Hence they must be identical."

† Read "Ninth line from the bottom of the page."

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THE MATHEMATICAL ASSOCIATION OF AMERICA

The Mathematical Association of America is devoted primarily to the interests of collegiate mathematics. It now has over two thousand individual and institutional members. There are twenty sections at present, representing thirty-one different States. The Association holds two national meetings per year and the sections hold one or two meetings per year. All meetings, both national and sectional, are reported in the official Journal.

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RARE BOOKS AND JOURNALS FOR SALE

Copies of Journals and Books mentioned below are available with me and are offered for sale. Those wishing to buy them will kindly write to me making their best offer for the same:—

I. *Progress Reports of the Indian Mathematical Club* Nos. 7 and 8.

Journal of the Indian Mathematical Club, Cloth Bound volumes for 1909, 1910.

Journal of the Indian Mathematical Society, Cloth Bound, 1911 to 1931, and loose issues of 1932 except the April issue.

The Mathematics Student, Vol. I, all the 4 issues loose.

All the journals are in good condition.

II. *Mathematical Questions and Solutions* reprinted from the Educational Times, Vols. 1 to 75 and New Series Volumes 1 to 29. Both sets worm-eaten but textual matter not seriously damaged.

III. COLEBROOK: *Algebra of the Hindus* (Algebra with Arithmetic and mensuration from Samskrit of Brhmagupta and Bhaskarar) London 1817. In original binding. Condition good.

K. F. VAKIL,

Sanjana Bungalow, Bhavnagar, India.

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